Contents

Pure Mathematics

Algebraic series

Binomial expansion:

$(a+b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \binom{n}{3} a^{n-3}b^3 + \ldots + b^n$, where n is a positive integer and $\binom{n}{r} = \dfrac{n!}{r!(n-r)!}$

Maclaurin expansion:

$$\mathrm{f}(x) = \mathrm{f}(0) + x\,\mathrm{f}'(0) + \frac{x^2}{2!}\mathrm{f}''(0) + \ldots + \frac{x^n}{n!}\mathrm{f}^{(n)}(0) + \ldots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots + \frac{n(n-1)\ldots(n-r+1)}{r!}x^r + \ldots \qquad (|x|<1)$$

$$\mathrm{e}^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^r}{r!} + \ldots \qquad (\text{all } x)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots + \frac{(-1)^r x^{2r+1}}{(2r+1)!} + \ldots \qquad (\text{all } x)$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots + \frac{(-1)^r x^{2r}}{(2r)!} + \ldots \qquad (\text{all } x)$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots + \frac{(-1)^{r+1} x^r}{r} + \ldots \qquad (-1 < x \le 1)$$

Partial fractions decomposition

Non-repeated linear factors:

$$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)}$$

Repeated linear factors:

$$\frac{px^2+qx+r}{(ax+b)(cx+d)^2} = \frac{A}{(ax+b)} + \frac{B}{(cx+d)} + \frac{C}{(cx+d)^2}$$

Non-repeated quadratic factor:

$$\frac{px^2+qx+r}{(ax+b)(x^2+c^2)} = \frac{A}{(ax+b)} + \frac{Bx+C}{(x^2+c^2)}$$

Probability and Statistics

Standard discrete distributions

Distribution of X	$\mathrm{P}(X=x)$	Mean	Variance
Binomial $\mathrm{B}(n,p)$	$\binom{n}{x} p^x (1-p)^{n-x}$	np	$np(1-p)$
Poisson $\mathrm{Po}(\lambda)$	$\mathrm{e}^{-\lambda}\dfrac{\lambda^x}{x!}$	λ	λ
Geometric $\mathrm{Geo}(p)$	$(1-p)^{x-1}p$	$\dfrac{1}{p}$	$\dfrac{1-p}{p^2}$

Standard continuous distribution

Distribution of X	p.d.f.	Mean	Variance
Exponential	$\lambda \mathrm{e}^{-\lambda x}$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$

Sampling and testing

Unbiased estimate of population variance:

$$s^2 = \frac{n}{n-1}\left(\frac{\Sigma(x-\bar{x})^2}{n}\right) = \frac{1}{n-1}\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right)$$

Unbiased estimate of common population variance from two samples:

$$s^2 = \frac{\Sigma(x_1-\bar{x}_1)^2 + \Sigma(x_2-\bar{x}_2)^2}{n_1+n_2-2}$$

Regression and correlation

Estimated product moment correlation coefficient:

$$r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\{\Sigma(x-\bar{x})^2\}\{\Sigma(y-\bar{y})^2\}}} = \frac{\Sigma xy - \dfrac{\Sigma x \Sigma y}{n}}{\sqrt{\left(\Sigma x^2 - \dfrac{(\Sigma x)^2}{n}\right)\left(\Sigma y^2 - \dfrac{(\Sigma y)^2}{n}\right)}}$$

Estimated regression line of y on x:

$$y - \bar{y} = b(x-\bar{x}), \quad \text{where } b = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\Sigma(x-\bar{x})^2}$$

Adapted from MF26 Formulae Sheet

Chapter 1 Functions

Beginner

1. *This question is about range of functions.*

Find the exact range of the following functions,

(a) $f(x) = x(x-4),\ x \geq 0;$ [2]

(b) $f(x) = x(x-4),\ 0 \leq x \leq 4;$ [2]

(c) $f(x) = \ln(x+3),\ x > -3;$ [2]

(d) $f(x) = \ln(x+3),\ 1 \leq x < 3;$ [2]

(e) $f(x) = e^x + 2,\ -3 < x < 3.$ [2]

2. *This question is about inverse functions.*

(i) For the following functions, find the inverse function, f^{-1}, giving its rule, domain and range.

(ii) Sketch, on the same diagram the graphs of $f(x)$ and its inverse, showing the relationship between them.

(a) $f(x) = x(x-4),\ x \leq -4;$ [4]

(b) $f(x) = \ln(x+3),\ x > -3;$ [4]

(c) $f(x) = e^x + 2,\ x < -3.$ [4]

3. *This question is about composite functions.*

The functions f, g, and h are given as follows:

$$f : x \mapsto 1 - e^{-2x},\ x \in \mathbb{R}^+,$$

$$g : x \mapsto \ln(3+2x),\ x \in \mathbb{R}^+,$$

$$h : x \mapsto 4 + \sin(\pi x),\ x \in \mathbb{R}^+.$$

(i) Find the ranges of $f(x)$, $g(x)$ and $h(x)$. [6]

(ii) Find the ranges of the composite function $f \circ g$ and $h \circ f$. [3]

4. *This question is about inverse and composite functions.*

The functions g and h are given as follows:

$$g : x \mapsto 3 - x^2,\ x \in \mathbb{R},\ x < k,$$

$$h : x \mapsto \ln(3-x),\ x \in \mathbb{R},\ x < 3.$$

(i) Find the largest integer value of k given that the inverse g^{-1} exists, and define g^{-1} in similar form. [4]

(ii) Using the k found in **(i)**, show that hg exists and find its corresponding range. [3]

5. *This question is about composite functions.*

The functions f, g, and h are given as follows:

$$f : x \mapsto -4x^2, x \in \mathbb{R},$$

$$g : x \mapsto e^{3x}, x \in \mathbb{R},$$

$$h : x \mapsto \ln(1-3x),\ x \in \mathbb{R},\ x < 0.25.$$

Determine whether the composite functions gf, fh, hf and hg exist. [6]

If the composite function exists, define the function in similar form and state its exact range. [4]

6. *This question is about inverse functions.*

The function f is given as follows:

$$f : x \mapsto \frac{1}{ax},\ x \geq \frac{1}{a},$$

where a is a constant greater than or equals to 1. Solve the equation $f(x) = f^{-1}(x)$. [4]

1. *This question is about piece wise functions.*

The function f is given as follows:

$$\mathrm{f}(x)=\begin{cases}7-2x, & 0\le x<3,\\ (2x-5)^2, & 3\le x<4.\end{cases}$$

and it is given that $\mathrm{f}(x-4)=\mathrm{f}(x)$ for all real values of x.

(i) State a reason why f does not have an inverse. [1]

(ii) Sketch the graph of $y=\mathrm{f}(x)$ for $-2<x\le 7$. [3]

(iii) Evaluate $\mathrm{f}(2019)$. [1]

2. *This question is about inverse and composite functions.*

The function f is given as follows:

$$\mathrm{f}:x\mapsto \frac{a}{(2x-4)^2},x\in\mathbb{R},x\ne 2.$$

where a is a positive constant.

(i) State the largest possible domain of f in the form of $(-\infty,b)$ such that the inverse function of f exists, where b is to be determined. [1]

(ii) Hence, define f^{-1} in a similar form, in terms of a. [3]

The functions g and h are given as follows:

$$\mathrm{g}:x\mapsto \ln(2x+3),x\in(-1,1],$$

$$\mathrm{h}:x\mapsto x^3-2x-1,x\in\mathbb{R}^+.$$

(iii) Verify whether the composite function gh exists. [2]

(iv) Find the rule and domain of the composite function hg, and hence find its range. [3]

3. *This question is about inverse and composite functions.*

The functions u and v are given as follows:

$$\mathrm{u}:x\mapsto \log_a x,x\in\mathbb{R}^+,$$

where a is a constant greater than 1,

$$\mathrm{v}:x\mapsto \frac{1}{x},x\in\mathbb{R}^+.$$

(i) State the ranges of u and v, and show that if t denotes the composite function $\mathrm{u}\circ\mathrm{v}$, then $\mathrm{t}(x)+\mathrm{u}(x)=0$. [3]

(ii) Explain briefly why the composite function $\mathrm{v}\circ\mathrm{u}$ cannot be properly defined unless the domain of u is restricted to a subset of $\mathbb{R}^+$, and state the largest possible subset which would be suitable for you to be defined. [2]

(iii) Define the inverses of u and v in similar form, and determine whether or not, $\mathrm{t}^{-1}(x)+\mathrm{u}^{-1}(x)=0$ [4]

4. *This question is about inverse and composite functions.*

The functions f and g are given as follows:

$$\mathrm{f}:x\mapsto \frac{ax}{2x-2},x\in\mathbb{R},x>1,$$

where a is a constant greater than 4,

$$\mathrm{g}:x\mapsto e^{2x^2-4},x\in\mathbb{R}.$$

(i) Sketch the graph of $y=\mathrm{g}(x)$, labelling any axial intercepts. [2]

(ii) If the domain of g is further restricted to $[b,\infty)$, state with a reason the least value of b for which the function g^{-1} exists. [2]

Advanced

1. *This question is about inverse functions and piece wise functions.*

The function f is given as follows:

$$\mathrm{f}: x \mapsto \begin{cases} -1-2x & \text{for } -3 < x \le -1, \\ -x^3 & \text{for } -1 < x \le 1, \\ 2x+1 & \text{for } 1 < x \le 3. \end{cases}$$

(i) State the range of the function and sketch the graph of $y = \mathrm{f}(x)$. [3]

(ii) Define the inverse function in similar form and determine whether this inverse function is a well-defined function. [5]

2. *This question is about inverse and composite functions.*

The function f is given as follows:

$$\mathrm{f}: x \mapsto |\ln(2x+8)|, x \in \mathbb{R}, -4 < x \le a,$$

where a is a real constant.

(i) Explain why f^{-1} does not exist when $a = 0$. [2]

(ii) State the maximum exact value of a such that the inverse function of f exists. [1]

Let a be the value found in **(ii)** for the rest of this question.

(iii) Define the inverse function of f in a similar form. [2]

(iv) The function g is defined by

$$\mathrm{g}: x \mapsto b + e^{2x}, x \ge -\ln 2,$$

where b is a real constant. Show that the composite function gf exists and determine the exact range of gf in terms of b. [3]

3. *This question is about inverse, composite and piece wise functions.*

The function f is given as follows:

$$\mathrm{f}: x \mapsto 3x^2 + 12x - 5, x \le a, a \in \mathbb{R}.$$

(i) Find algebraically, the largest integer value of a such that the inverse function of f exists. [2]

For this value of a, define the inverse function of f in similar form. [3]

Another function g is defined by

$$\mathrm{g}: x \mapsto \begin{cases} 3-2x^2, & \text{for } 0 < x \le 2, \\ -x-3, & \text{for } 2 < x \le 4, \end{cases}$$

and that $\mathrm{g}(x) = \mathrm{g}(x+4)$ for all real values of x.

(ii) Sketch the graph of $y = \mathrm{g}(x)$ for $-3 < x \le 5$. [3]

(iii) Using the previous results, explain why composite function $\mathrm{f}^{-1}\mathrm{g}(x)$ exists and find the exact value of $\mathrm{f}^{-1}\mathrm{g}(6)$ [3]

4. *This question is about inverse and composite functions.*

The function f is given as follows:

$$\mathrm{f}: x \mapsto a + \frac{3}{1-x}, x \in \mathbb{R}, x \ne 1,$$

where a is a negative real number.

(i) By differentiating $\mathrm{f}(x)$, show that f^{-1} exists. [2]

(ii) Find the set of values of a such that the equation $\mathrm{f}(x) = 2x$ has real solutions. [3]

The functions g and h are given as follows:

$$\mathrm{g}: x \mapsto \mathrm{f}(x), x < 0,$$

$$\mathrm{h}: x \mapsto \left[3x - \left(a + \frac{2}{3}\right)\right]^2, x \in \mathbb{R}.$$

(iii) Find the range of hg exactly, in terms of a. [2]

5. *This question is about inverse and composite functions.*

The functions g and h are given as follows:

$$\mathrm{g}: x \mapsto 1 + \frac{2}{x-a}, x \in \mathbb{R}, x < a, a \ge 1,$$

$$\mathrm{h}: x \mapsto \ln x^2, x \in \mathbb{R}, 0 < x < 1.$$